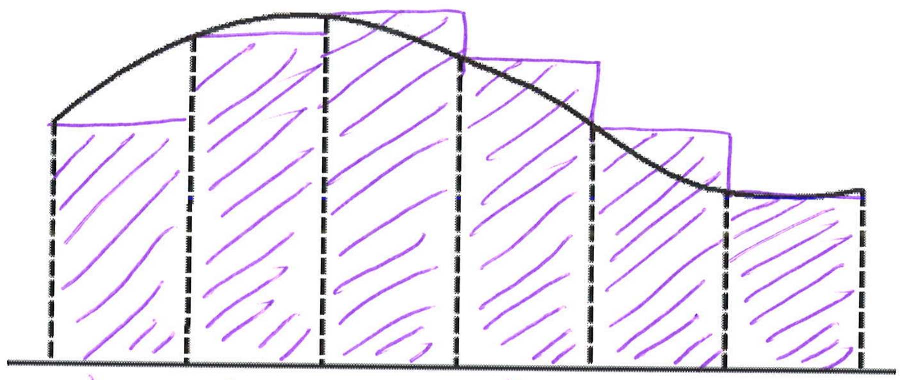
Writing a Riemann Sum using Σ-notation

Example 1: Suppose that f is a function. We want to write a left Riemann sum for f on the interval [1,3]. We will use 6 subintervals of equal length.



$1 \xrightarrow{\frac{1}{3}} \frac{4}{3} \xrightarrow{1/3} \frac{5}{3} \xrightarrow{1/3} 2 \xrightarrow{1/3} \frac{7}{3} \quad \frac{8}{3} \quad 3$

$1 \longrightarrow 1+\frac{1}{3} \longrightarrow 1+2(\frac{1}{3}) \longrightarrow 1+3(\frac{1}{3})$

$$L_6 = f(1)\left(\tfrac{1}{3}\right) + f(4/3)\left(\tfrac{1}{3}\right) + f(5/3)\left(\tfrac{1}{3}\right) + f(2)\left(\tfrac{1}{3}\right) + f(7/3)\left(\tfrac{1}{3}\right)$$

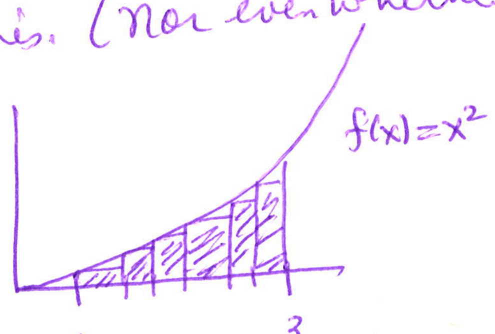
$$+ f(8/3)\left(\tfrac{1}{3}\right)$$

$$= \sum_{i=1}^{6} f\left(1+\frac{i-1}{3}\right)\left(\frac{1}{3}\right)$$

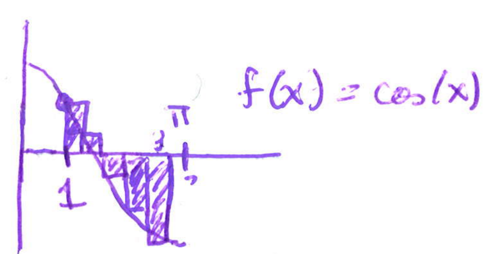
Pattern:

$i=1$	$1 = 3/3 = 1+\frac{0}{3}$
$i=2$	$\frac{4}{3} = 1+\frac{1}{3}$
$i=3$	$\frac{5}{3} = 1+2(\frac{1}{3})$
$i=4$	$2 = 6/3 = 1+3(\frac{1}{3})$
$i=5$	$7/3 = 1+4(\frac{1}{3})$
$i=6$	$8/3 = 1+5(\frac{1}{3})$

Notice: The shape of the graph of f didn't enter into this. (nor even whether it is always positive!)

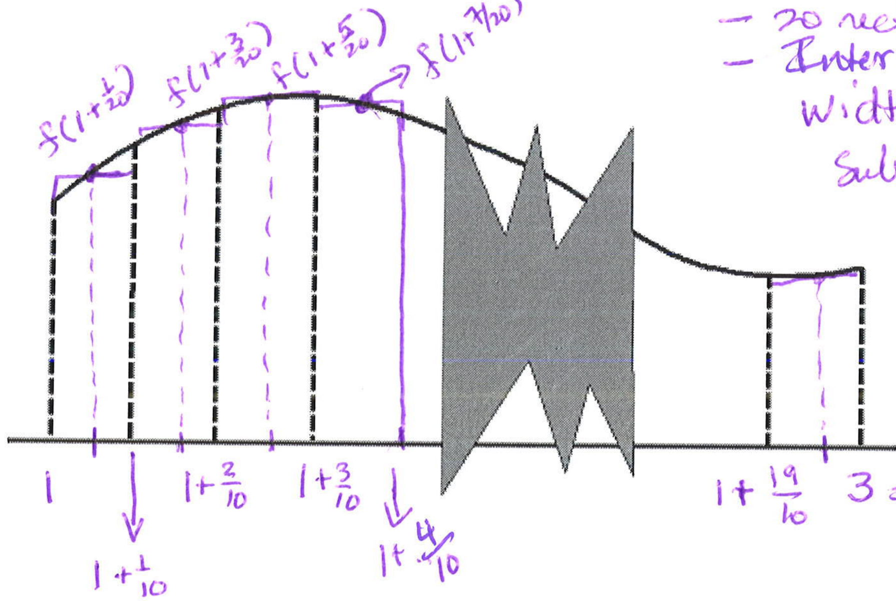


$$L_6 = \sum_{i=1}^{6} \left(1+\frac{i-1}{3}\right)^2\left(\frac{1}{3}\right)$$



$$L_6 = \sum_{i=1}^{6} \cos\left(1+\frac{i-1}{3}\right)\left(\frac{1}{3}\right)$$

Example 2: Suppose that f is a function. We want to write a midpoint Riemann sum for f on the interval [1,3]. We will use 20 subintervals of equal length.



- 20 rectangles in all
- Interval [1, 3].

width of each subinterval:

$$\frac{3-1}{20} = \frac{2}{20} = \frac{1}{10}$$

$$\text{Area of } 1^{st} \text{ rectangle} = \underbrace{\frac{1}{10}}_{\text{width}} \underbrace{f\left(1+\frac{1}{20}\right)}_{\text{height}}$$

$$\text{Area of } 2^{nd} \text{ rectangle} = \frac{1}{10} f\left(1+\frac{3}{20}\right)$$

$$\text{Area of } 3^{rd} \text{ rectangle} = \frac{1}{10} f\left(1+\frac{5}{20}\right)$$

$$\text{Area of } 4^{th} \text{ rectangle} = \frac{1}{10} f\left(1+\frac{7}{20}\right)$$

$$\text{Area of "}i^{th}\text{" rectangle} = \frac{1}{10} f\left(1+\frac{2i-1}{20}\right)$$

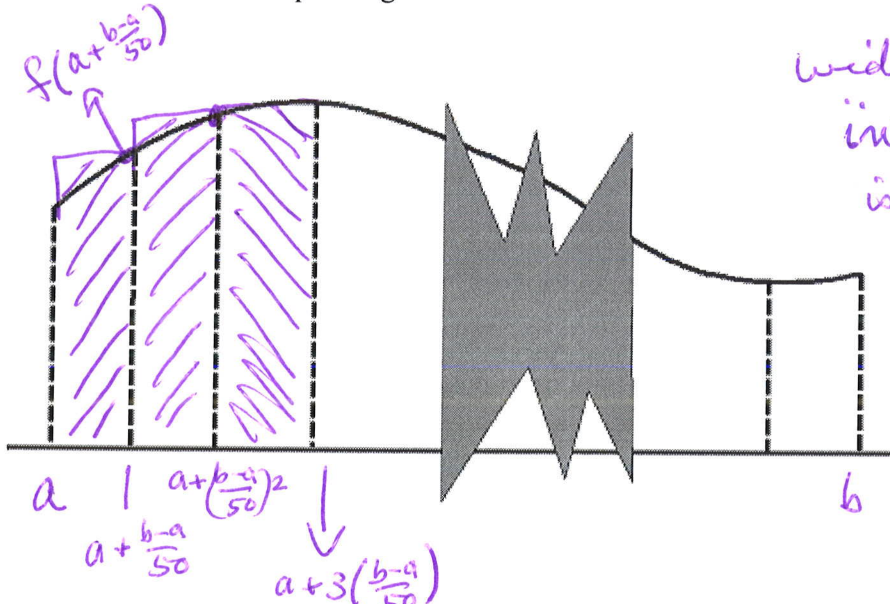
$$M_{20} = \sum_{i=1}^{20} \frac{1}{10} f\left(1+\frac{2i-1}{20}\right)$$

If our function had been $f(x) = x^3 - 2x$:

$$M_{20} = \sum_{i=1}^{20} \frac{1}{10}\left[\left(1+\frac{2i-1}{20}\right)^3 - 2\left(1+\frac{2i-1}{20}\right)\right]$$

$\boxed{\text{The shape of the graph doesn't matter! What does matter?}}$

Example 3: Suppose that f is a function. We want to write a right Riemann sum for f on the interval [a,b]. We will use 50 subintervals of equal length.



width: the interval [a, b] is divided up into 50 pieces of equal length. Each is thus $\frac{b-a}{50}$ in width.

$$\text{Area of } 1^{st} \text{ rectangle} = \underbrace{\left(\frac{b-a}{50}\right)}_{\text{width}} \overbrace{\left(f\left(a+\frac{b-a}{50}\right)\right)}^{\text{height}}$$

$$\text{Area of } 2^{nd} \text{ rectangle} = \left(\frac{b-a}{50}\right) f\left(a+2\left(\frac{b-a}{50}\right)\right)$$

$$\text{Area of } 3^{rd} \text{ rectangle} = \left(\frac{b-a}{50}\right) f\left(a+3\left(\frac{b-a}{50}\right)\right)$$

$$\text{area of } i^{th} \text{ rectangle} = \left(\frac{b-a}{50}\right) f\left(a+i\left(\frac{b-a}{50}\right)\right)$$

$$R_{50} = \sum_{i=1}^{50} \left(\frac{b-a}{50}\right) f\left(a+i\left(\frac{b-a}{50}\right)\right)$$

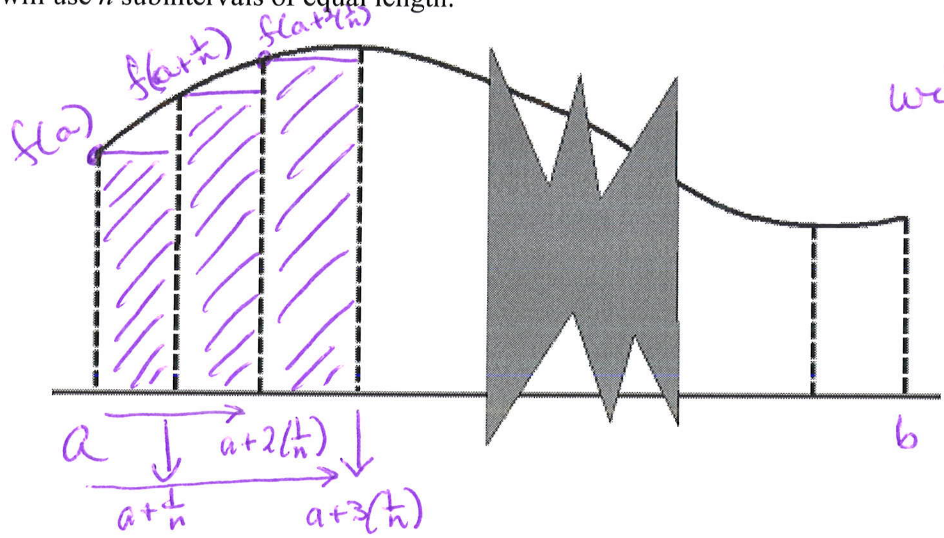
If $f(x) = e^{-x^2}$

$$R_{50} = \sum_{i=1}^{50} \left(\frac{b-a}{50}\right)\left(\exp\left(-\left(a+i\left(\frac{b-a}{50}\right)\right)^2\right)\right)$$

So what did affect this (if not the shape of the graph?)

- which interval we were working on
- whether we were using left, right or midpt Riemann sums.

Example 4: Suppose that f is a function. We want to write a left Riemann sum for f on the interval [a,b]. We will use n subintervals of equal length.



width $= \frac{b-a}{n}$

Area of 1^{st} rectangle $= \left(\frac{b-a}{n}\right) f\left(a + 0\left(\frac{1}{n}\right)\right)$

area of 2^{nd} rectangle $= \left(\frac{b-a}{n}\right) f\left(a + 1\left(\frac{1}{n}\right)\right)$

Area of 3^{rd} rectangle $= \left(\frac{b-a}{n}\right) f\left(a + 2\left(\frac{1}{n}\right)\right)$

Area of i^{th} rectangle $= \left(\frac{b-a}{n}\right) f\left(a + (i-1)\left(\frac{1}{n}\right)\right)$

$$L_n = \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + (i-1)\left(\frac{1}{n}\right)\right)$$

If $f(x) = 1 - x^2$ and $a = -2$ and $b = 12$ ...

$$L_n = \sum_{i=1}^{n} \left(\frac{14}{n}\right) f\left(-2 + (i-1)\left(\frac{1}{n}\right)\right)$$

$$= \sum_{i=1}^{n} \left(\frac{14}{n}\right) \left(1 - \left(-2 + (i-1)\left(\frac{1}{n}\right)\right)^2\right)$$